A REFORMULATION OF THE COSMIC RAY SOLAR

MODULATION PROBLEM

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ABSTRACT

The propagation of galactic cosmic rays in the interplanetary medium is examined and a transport equation for the cosmic ray energy spectrum is derived. By constructing a Boltzmann equation for the cosmic ray phase space distribution function and by taking velocity moments over directions in momentum space, the continuity and momentum conservation equations for each energy element of the cosmic ray gas are obtained. The effect of scattering (by magnetic irregularities) on the distribution function is treated by a Fokker Planck equation while the effect of scattering on the differential streaming velocity is treated by means of a relaxation approximation. By linearizing the momentum equation, we obtain expressions for the differential streaming velocity at an arbitrary point in the medium in terms of the local energy spectrum. The value of the first Fokker Planck coefficient is evaluated and found to represent adiabatic cooling of the cosmic ray gas while an additional deceleration arises from streaming along a polarization electric field. The resulting transport equation for the energy spectrum reduces to the equation Parker (1965) studies if the effects of the interplanetary polarization electric field are neglected.

INTRODUCTION

The detailed understanding of cosmic ray propagation in the interplanetary medium is currently of special importance because measurements on low energy cosmic rays, which are believed to be most sensitive to solar effects, are only now being widely carried out. The low energy data (e.g. Fan, etal. 1965) have revealed a number of unexpected properties of the cosmic ray energy and charge spectrum and as more data is collected in the present years approaching solar maximum it will become increasingly important to know the extent of the influence of solar modulation on the data.

A large amount of theoretical work has been devoted to the modulation problem and the present work is closely related to the calculations of Parker (1965), Axford (1965), and Quenby (1966), in particular. One of the aims of the present calculation is to indicate the relationship between these approaches to the modulation problem and to show that a generalization of Axford's technique leads to Parker's equation of motion for the energy spectrum if certain approximations are made.

In Axford's approach to the problem, the cosmic ray motion is separated into continuous flow (due to a steady electromagnetic field) and scattering (due to fluctuations in the fields) by means of a Boltzmann equation in which the collision term represents the cosmic ray scattering by field fluctuations. Then by taking moments of this equation with respect to $\bar{v}$ over all velocity space, Axford obtains equations for the cosmic ray number density and streaming velocity. These equations are then solved and give Parker's well known exponential modulation factor for the number density as well as general expressions for each component of the streaming velocity.

Throughout his entire analysis Axford was concerned only with the intensity and streaming of the total cosmic ray gas, irrespective of any effects of the modulation on the cosmic ray energy spectrum. In Quenby's work there is also discussion of the deceleration of cosmic rays as they diffuse throughout the solar system. This deceleration is pictured as arising from two phenomena. In the first place, the outward flow of the solar wind creates a polarization electric field and it may be shown that cosmic rays which drift due to the magnetic gradient in the interplanetary field also drift against the polarization electric field, and hence lose energy. Secondly the magnetic irregularities responsible for the cosmic ray scattering expand with the solar wind and this produces an overall Fermi deceleration of the cosmic rays as the rebound from the diverging irregularities. Quenby obtains deceleration rates due to these processes and then estimates the effect of this deceleration on the energy spectrum of the incoming cosmic rays.

In the present work we wish to treat the modulation due to deceleration in a more unified way and we shall do so in a manner suggested by Axford's work. Namely, starting from the same Boltzmann equation, we shall take moments of this equation with respect to $\vec{v}$ but only over directions in velocity space. In this way we are led to coupled transport equations for the cosmic ray energy spectrum and differential streaming velocity, rather than for the cosmic ray number density and streaming velocity. These equations therefore have built into them not only convection, diffusion and streaming but also, the cosmic ray energy loss due to the steady electromagnetic field and scattering.

We do not consider solutions for the energy spectrum however, and therefore are not proposing a specific modulation function at this time. Our discussion centers exclusively on the transport equation itself.

TRANSPORT EQUATIONS FOR THE ENERGY SPECTRUM AND STREAMING VELOCITY

The interplanetary space through which cosmic rays pass is known to possess a magnetic field which is being transported outward from the sun by the solar wind. To a first approximation this field has an appearance, as shown in Figure 1; and we shall make use of the following properties of the field in the present calculation. (1) The average magnetic field at any point $\bar{r}$ in the ecliptic plane is given by

$$B_r = B_o(r_o)\left(\frac{r_o}{r}\right)^2, \quad B_\theta = 0, \quad B_\phi = \frac{-\Omega r}{V_o} B_r$$

where Ω is the angular velocity of the sun, V_o is the solar wind speed and $B_o(r_o)$ is the solar magnetic field at the base of the field line passing through $\bar{r}$. (2) The above magnetic field is observed to lie in sectors or tubes of fixed magnetic polarity, and we shall further assume that $B_o(r_o)$ is constant within a given sector so that $\bar{B}(\bar{r})$ is static as long as $\bar{r}$ remains within that sector. (On a time scale of weeks, however, alternating sectors sweep past $\bar{r}$ so that $\bar{B}(\bar{r})$ has a time dependence which is strong during the passage of sector boundaries.) (3) The average solar wind blows radially, with uniform speed everywhere in the ecliptic and is only a weak function of time. The flow of the solar plasma past $\bar{r}$ in the presence of the above magnetic field produces a polarization electric field, $\bar{E} = -\bar{V}_o \times \bar{B}$, which is static as long as $\bar{r}$ remains within a given magnetic sector. However, the sense of this electric field reverses as a neighboring magnetic sector of opposite polarity sweeps past $\bar{r}$.

The above description is an idealization which applies only to the average, quiet interplanetary electromagnetic field; the true fields are found to be much more complicated. There are variations in magnetic field strength and solar wind speed over the solar surface and there are instabilities in the plasma which produce waves or fluctuations which are superimposed on the quiet electromagnetic field. In fact, the power spectrum of these waves in the spiral magnetic field has already been measured. (e. g. Holzer etal. 1966).

We are now interested in the propagation of cosmic rays through the above electromagnetic field and shall specialize our present discussion to particles with 10 MeV < E < 200 MeV, so that we can keep the analysis non-relativistic. The generalization to relativistic particles is simple and can be carried out later. Particles in the above energy range have gyration radii small compared with the scale size of the unperturbed magnetic field and therefore a particular cosmic ray particle will guide adiabatically along $\bar{B}(\bar{r})$ and will drift slowly normal to $\bar{B}$, with velocity,

$$\bar{V}_D = \frac{\bar{E} \times \bar{B}}{B^2} + \frac{\bar{B}}{B^2} \times [\mu \nabla |B| + m\ddot{\bar{R}}]$$

where μ is the particle orbital magnetic moment and $\ddot{\bar{R}}$ is the acceleration of its guiding center. There are two aspects of this adiabatic motion which play an important role in the modulation of galactic cosmic rays. (1) Far from the sun where the magnetic field is essentially azimuthal, the electric drift carries particles radially outward at the solar wind velocity. (2) The combined gradient and curvature drift moves particles in a polar direction with a drift velocity essentially proportional to the kinetic energy of the particle. Furthermore, this drift velocity is always in a direction such that

$$\frac{d\mathcal{E}}{dt} = \bar{E} \cdot \bar{V}_D < 0$$

and thus, all drifting particles (whether electrons or nuclei, or whether $\overline{B}$ is directed toward or away from the sun) lose energy. The unperturbed interplanetary field therefore has the two properties of carrying particles out of the solar system as well as decelerating them.

The total electromagnetic field is of course the unperturbed field plus the fluctuations, $\delta\overline{B}$ and $\delta\overline{E}$ and the fluctuations give rise to forces which displace or scatter particles out of their idealized helices. Furthermore, the fluctuations responsible for the scattering tend to be carried outward by the solar wind and therefore comprise an expanding magnetic configuration from which the particle scatters. The particle therefore suffers a Fermi deceleration which is distinct from the deceleration discussed above. Perturbations in the interplanetary field therefore have the properties of scattering and cooling the particles.

We now wish to consider a distribution of cosmic rays of a fixed species having velocity $\overline{v}$ at the position $\overline{r}$ at time t; $f(\overline{r},\overline{v},t)$. This distribution function evolves in time due to the steady unperturbed Lorentz force,

$$\overline{F} = q(\overline{v} \times \overline{B} + \overline{E})$$

and due to the scattering imparted by the irregularities, $\delta\overline{F}$. The continuity of particle flow in phase space is expressed by the Boltzmann equation,

$$\frac{\partial f}{\partial t}(x_i, v_i t) + v_i \frac{\partial f}{\partial x_i} + \frac{F_i}{m} \frac{\partial f}{\partial v_i} = \left.\frac{\partial f}{\partial t}\right)_c \qquad (1)$$

where $\partial f/\partial t)_c$ is the rate at which particles are scattered into the volume element at (x_i, v_i) by magnetic irregularities. The distribution function f is more detailed than necessary for our present purposes because cosmic ray measurements generally give only a speed (or energy) distribution of the particles. That is, the quantity which is observed and which shall be of interest to us here will be

$$h(x_i, v, t) \equiv \int f(x_i, v_i, t) v^2 d\Omega' \tag{2}$$

where the integration is over all directions on the shell v = constant in velocity space. The quantity h is essentially the cosmic ray energy spectrum at x_i at the time, t. The average value of any function of the velocity $A(v_i)$ averaged over all particles of speed, $|v_i|$ is then

$$\langle A \rangle = \frac{1}{h} \int A(v_i) f(x_i v_i t) v^2 d\Omega' \tag{3}$$

In order to obtain the transport equation satisfied by h, one may integrate equation 1 term by term over the shell, v = constant. This integration can be carried out (see Appendix A) and gives

$$\frac{\partial}{\partial t} h(x_i vt) + \frac{\partial}{\partial x_i}(h \langle v_i \rangle) + \frac{qE_i}{m} \frac{\partial}{\partial v} \left(\frac{h \langle v_i \rangle}{v}\right) = \partial h/\partial t)_c \tag{4}$$

where $\langle v_i \rangle$ is the average (or streaming) velocity of particles with speed v (hereafter called the differential streaming velocity) and $\partial h/\partial t)_c$ is the rate at which cosmic rays are scattered into the speed v at x_i. This equation merely states the continuity of particle flow in (x_i, v) space. It is important to note the difference between the above continuity equation and the usual continuity equation in coordinate space for the particle number density. The last two terms, which are absent in the usual continuity equation, indicate the fact that there is a source of particles with speed v at x_i. Specifically, the steady electric field as well as scattering can produce particles of speed v from particles at other speeds.

Although equation 4 is a transport equation for h which suitably contains all the propagation and deceleration effects due to $\bar{F}$ and $\delta\bar{F}$, it also contains the differential streaming velocity which at present is an undetermined quantity. We therefore wish to obtain an equation of motion for the differential streaming velocity, and this may be done by taking the moment of the velocity v_j over the spherical shell v = constant in velocity space, for each term of the Boltzmann equation. Defining the random velocity w_i by $w_i = v_i - \langle v_i \rangle$, the resulting equation is (see Appendix A)

$$h\left[\frac{\partial\langle v_j\rangle}{\partial t} + \langle v_i\rangle\frac{\partial\langle v_j\rangle}{\partial x_i} + \frac{qE_i}{mv}\langle v_i\rangle\frac{\partial\langle v_j\rangle}{\partial v}\right] = \frac{h}{m}\langle F_j\rangle - \frac{\partial}{\partial x_i}S^{ij}_{(v)} - \frac{qE_i}{m}\frac{\partial}{\partial v}\left(\frac{S^{ij}}{v}\right)$$

$$- \langle v_j\rangle\, \partial h/\partial t)_c + P_j \qquad (5)$$

where

$$S^{ij}_{(v)} \equiv h\langle w^i w^j\rangle$$

is the stress tensor associated with the random motion of particles with speed v. Also,

$$P_j \equiv \int v_j\left.\frac{\partial f}{\partial t}\right)_c v^2 d\Omega'$$

is the rate at which scattering increases the momentum of particles with speed v. Equation 5 is a dynamical equation which states that an acceleration of the ($x_i \to x_i + \Delta x_i$, $v \to v + \Delta v$) element of the cosmic ray gas results from (1) the presence of applied forces, (2) a net flow of momentum into the ($\Delta x_i, \Delta v$) element and (3) a net scattering of momentum into the ($\Delta x_i, \Delta v$) element of the cosmic ray gas. We now have an equation for $\langle v_j \rangle$, as desired, but the equation involves an undetermined stress tensor which is second order in the velocity. Taking further velocity moments of the Boltzmann equation only produces a heirarchy of equations which involve successively higher order correlations in the particle velocities, and therefore a closed set of equations for $h, \langle v_j \rangle$, S^{ij}, etc. cannot be obtained in this way. Instead we shall make some simplifying approximations which make equations 4 and 5 tractable.

We first examine the random velocity stress tensor. The cosmic ray gas as a whole is observed to be nearly isotropic; that is the streaming velocity,

$$nu_i \equiv \int v_j f(x_i v_i t) d^3v = \int \langle v_j \rangle h(v) dv$$

nearly vanishes. The differential streaming velocity however can not in general be zero in the steady state since, from equation 4

$$\frac{\partial}{\partial x_i} (h \langle v_i \rangle) \neq 0$$

Nevertheless, we shall assume that particles at a particular v are not far from isotropic so that S^{ij} may be represented as diagonal and having equal components:

$$S^{ij} = h \langle w^i w^j \rangle = \frac{1}{3} h \langle w^2 \rangle \delta^{ij} = \frac{h}{3} [v^2 - \langle v \rangle^2] \delta^{ij} \qquad (6)$$

Furthermore since the differential streaming velocity will be small compared with the particle velocity, our approximation becomes

$$S^{ij} \simeq \frac{hv^2}{3} \delta^{ij} \tag{7}$$

We now wish to simplify the collision terms apprearing in equations 4 and 5. Let us first examine equation 4. This equation gives the net rate at which cosmic rays accumulate at $(x_i v)$ due to the simultaneous action of applied forces, spatial flow and scattering. We may focus our attention on scattering alone by "switching off" the steady forces and gradients; that is, let us consider a uniform field-free region. In this case particles accumulate at $(x_i v)$ due to scattering alone. If we assume that the particles are scattered in a completely random manner by the $\delta\bar{F}$'s then our field-free, gradient-free distribution should satisfy (see Montgomery and Tidman, 1964)

$$h'(x_i,v,t) = \int h'(x_i,\ v - \Delta v,\ t - \Delta t)\ P(v - \Delta v,v)d\Delta v \tag{8}$$

where $P(v - \Delta v,\Delta v)$ is the transition probability for scattering from $v - \Delta v$ to v in the fixed time, Δt where Δt includes many scatterings. If the integrand is expanded in a power series in Δv and Δt about (v,t) then one obtains the Fokker-Planck expansion,

$$\frac{\partial h'}{\partial t} = - \frac{\partial}{\partial v} \left[\frac{\langle\Delta v\rangle}{\Delta t} h\right] + \frac{1}{2} \frac{\partial^2}{\partial v^2} \left[\frac{\langle\Delta v^2\rangle}{\Delta t} h\right] + \cdots \tag{9}$$

$$\langle(\Delta v)^n\rangle \equiv \int \Delta v^n P d\Delta v$$

We shall later demonstrate that only the first term is important. Then associating $\frac{\partial h'}{dt}$ with $\left.\frac{\partial h}{\partial t}\right)_-$, equation 4 may be rewritten

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_i} (h \langle v_i \rangle) + \frac{\partial}{\partial v} [h (\frac{qE_i \langle v_i \rangle}{mv} + \frac{\langle \Delta v \rangle}{\Delta t})] = 0 \tag{10}$$

Let us next apply an analagous approach to equation 5. This equation gives the differential streaming velocity which results from the simultaneous action of applied forces, gradients and scattering, and once again we can isolate the effects of scattering by considering a field-free, gradient-free configuration in which the transient response of $\langle v_j \rangle$ is given by

$$h' \frac{\partial \langle v_j \rangle'}{\partial t} = P_j' - \langle v_j \rangle' \left.\frac{\partial h}{\partial t}\right)_c \tag{11}$$

If we now recall certain basic properties of our interplanetary model, we can approximate the solution to the above equation. Since the scattering is caused by a distribution of randomly oriented magnetic irregularities, we may assume that the deflections produced in a cosmic ray trajectory are isotropic in the rest frame of the irregularity. However since the irregularities move radially outward with the solar wind speed they will transfer a net outward momentum to the particles, and the particle differential streaming velocity will gradually <u>relax</u> from its initial value to the solar wind velocity:

$$(\langle v_j' \rangle - V_o)\Big|_t = (\langle v_j' \rangle - V_o)\Big|_o \exp\left(\frac{-t}{\tau(x_i, v)}\right), \text{ or } \frac{\partial \langle v_j' \rangle}{\partial t} = -\frac{\langle v_j' \rangle - V_o}{\tau(x_i, v)} \tag{12}$$

where $\tau(x_i, v)$ is the relaxation time or characteristic time needed for the δF's to effectively scatter particles of speed v at x_i. We have thus introduced τ in a phenomenological way as a relaxation time for the differential streaming velocity. Jokipii (1966) and Roelof (1966) have also obtained a relaxation time but in a more fundamental way from a statistical description of the particle motion through the irregularities. Using the result of our relaxation approximation in equation 11, we obtain

$$P_j - \langle v_j \rangle \left.\frac{\partial h}{\partial t}\right)_c = - \frac{\langle v_j \rangle - V_o}{\tau\,(x_i v)} h$$

as the scattering contribution to $\langle v_j \rangle$. Equation 5 then becomes

$$h \left[\frac{\partial \langle v_j \rangle}{\partial t} + \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} + \frac{qE_i}{mv} \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial v} \right] = \frac{h}{m} \langle F_j \rangle - \frac{v^2}{3} \frac{\partial h}{\partial x_j} - \frac{qE_j}{m} \frac{\partial}{\partial v} \left(\frac{hv}{3}\right) - \frac{h(\langle v_j \rangle - V_o)}{\tau\,(x_i v)} \tag{13}$$

where use has also been made of equation 7. The preceding approximations have therefore led us to two equations (equations 10 and 13) which involve only the energy spectrum h and the differential streaming velocity $\langle v_j \rangle$. Once the relaxation function τ and the Fokker-Planck coefficient $\langle \Delta v \rangle / \Delta t$ are evaluated, these equations are in principle solvable.

PROPERTIES OF THE TRANSPORT EQUATIONS

In order to simplify equation 13 we shall again make use of the fact that the differential streaming velocity is small, and we shall accordingly neglect the quadratic terms in equation 13. We are interested in steady state solutions and therefore the linearized momentum equation becomes

$$0 = -\frac{v^2}{3}\frac{\partial h}{\partial x_j} - \frac{qE_j}{m}\frac{\partial}{\partial v}\left(\frac{hv}{3}\right) + \frac{qh}{m}\left(E_j + [\langle v\rangle \times B]_j\right) + \frac{h}{\tau}(V_o - \langle v_j\rangle) \qquad (14)$$

Equation 14 comprises three algebraic equations and recalling the forms of $\bar{V}_o$, $\bar{E}(\bar{r})$ and $\bar{B}(\bar{r})$ these equations may be solved simultaneously for $\langle v_j\rangle$. We have already assumed azimuthal uniformity within a given magnetic sector and therefore $\partial/\partial\phi = 0$. The solutions are easily found to be

$$h\langle v_r\rangle = -(1+\omega_r^2\tau^2)K_\perp\frac{\partial h}{\partial r} + \omega_\phi\tau K_\perp\frac{\partial h}{r\partial\theta} - \omega_\phi^2\tau^2\frac{V_o}{1+\omega^2\tau^2}\frac{\partial}{\partial v}\left(\frac{hv}{3}\right) + hV_o \qquad (15a)$$

$$h\langle v_\theta\rangle = \omega_\phi\tau K_\perp\frac{\partial h}{\partial r} - K_\perp\frac{\partial h}{r\partial\theta} + \frac{\omega_\phi\tau V_o}{1+\omega^2\tau^2}\frac{\partial}{\partial v}\left(\frac{hv}{3}\right) \qquad (15b)$$

$$h\langle v_\phi\rangle = \omega_\phi\omega_r\tau^2K_\perp\frac{\partial h}{\partial r} + \omega_r\tau K_\perp\frac{\partial h}{r\partial\theta} - \frac{\omega_\phi\omega_r\tau^2}{1+\omega^2\tau^2}V_o\frac{\partial}{\partial v}\left(\frac{hv}{3}\right) \qquad (15c)$$

where $\omega_i = \dfrac{q|B_i|}{m}$, $K_\perp = \dfrac{v^2\tau}{3(1+\omega^2\tau^2)}$. With the definitions,

$$K_{ij} = K_\perp\begin{pmatrix} 1+\omega_r^2\tau^2 & -\omega_\phi\tau & -\omega_\phi\omega_r\tau^2 \\ -\omega_\phi\tau & 1 & -\omega_r\tau \\ -\omega_\phi\omega_r\tau^2 & -\omega_r\tau & 1+\omega_\phi^2\tau^2 \end{pmatrix}$$

$$C_j = \frac{\omega_\phi \tau V_o \frac{\partial}{\partial v}\left(\frac{hv}{3}\right)}{1 + \omega^2 \tau^2}(\omega_\phi \tau,\ -1,\ \omega_r \tau)$$

equation 15 may be written

$$h\langle v_j \rangle = hV_o \delta^{jr} - K_{ij} \frac{\partial h}{\partial x_i} - C_j \tag{16}$$

The coefficient $K_{ij}(\bar{r})$ is seen to be a spatial diffusion tensor for particles scattering from irregularities in the spiral magnetic field and C_j is a flow vector which arises from deceleration of particles in the steady field, $\bar{E}(\bar{r})$. Equation 16 thus determines the differential streaming velocity at a point $\bar{r}$ in terms of the local energy spectrum.

If equation 16 is substituted into the continuity relation, equation 10, one obtains

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_j}\left[hV_o \delta_{jr} - K_{ij} \frac{\partial h}{\partial x_i} \right] - \frac{\partial C_i}{\partial x_j} + \frac{\partial}{\partial v}\left[h \left(\frac{\langle \Delta v \rangle}{\Delta t} + \frac{qE_i}{mv} \langle v_i \rangle \right) \right] = 0 \tag{17}$$

as the transport (or Fokker-Planck) equation for the cosmic ray energy spectrum. In Appendix B, it is demonstrated that the scattering of cosmic rays in the expanding solar wind leads to an adiabatic cooling, and as long as the particle speed is large compared with the solar wind speed, the kinetic energy also decreases adiabatically. This was first pointed out by Parker (1963). Hence,

$$\frac{\langle \Delta v \rangle}{\Delta t} = -\frac{2v}{3r} V_o$$

With this value of the Fokker-Planck coefficient, equation 17 differs from the equation Parker (1965) studies only by the additional terms,

$$\frac{\partial}{\partial v}\left[h \frac{qE_i \langle v_i \rangle}{mv} \right] - \frac{\partial C_i}{\partial x_j}$$

Since both terms vanish if $E_i = 0$, it appears that our transport equation reduces to Parker's equation in the absence of a steady electric field in the interplanetary medium. It is not clear however that neglecting the electric field is justifiable. Only in the case that the spiral magnetic field is completely obliterated by the irregularities would the steady electric field vanish. In this case we would also expect $\omega\tau \ll 1$ so that the particle diffusion is isotropic. Our analysis, on the otherhand, has anisotropic diffusion implicitly built in by our specification of a well defined large scale magnetic field in the ecliptic plane. To treat the case for which the spiral magnetic field is dominated by the magnetic irregularities using our formulation of the problem, we would let $|\delta B| \gg |B|$. Then $|E| = |V_o \times B|$ and $|C_j|$ would also be negligible and we would regain Parker's equation.

Equation 16 is an interesting result in its own right since it gives us each component of the streaming velocity and its energy dependence. Contrary to the total streaming velocity of the cosmic ray gas, none of the components of $\langle v_j \rangle$ vanishes, in general.

SUMMARY

In the present work we have been interested in the derivation and properties of the transport equation describing the variation of the cosmic ray energy spectrum within the interplanetary medium. The fundamental properties of the interplanetary medium which determine the modulation are (1) the underlying spiral magnetic field, (2) its associated electric field, (3) the irregularities in the magnetic field and (4) the flow properties of the solar wind. The combined effects of the spiral magnetic field and its irregularities cause particles to partially guide along and diffuse across magnetic field lines. The scattering by irregularities can be characterized by a relaxation time which is defined as the time needed for scattering to isotropize particles of a given energy, and such a description of the scattering implies that particles undergo diffusion according to a diffusion tensor which is determined by the relaxation time and the spiral magnetic field. A prescription for calculating this relaxation time from the power spectrum of the magnetic irregularities has been given by Jokipii (1966) and Roelof (1966). By discussing the diffusion in terms of a relaxation time instead of a "mean free time" it becomes possible to discard the unrealistic picture of isolated scattering centers separated by an energy dependent mean free path. The electric field normal to the ecliptic gives rise to particle deceleration and this deceleration modulates the energy spectrum if there is streaming along the electric field direction. The scattering from magnetic irregularities also gives rise to a deceleration which is

literally an adiabatic cooling of the particles.

The transport equation we derive gives the combined effect of all the above processes on the cosmic ray energy spectrum and our equation reduces to Parker's equation (Parker, 1965) in the absence of a steady electric field. Our analysis also determines the differential streaming velocity of cosmic rays at any point in the ecliptic in terms of the local energy spectrum.

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FIGURE CAPTIONS

Figure 1. Interplanetary Field Model. The sector structure of the average magnetic field is illustrated by means of two sectors of opposite polarity. The electric field is out of the ecliptic in one sector and into the ecliptic in the other.

Figure 2. One Dimensional Model of Adiabatic Cooling. Magnetic irregularities are represented by the artifice of pistons of area A located at $x_{-n} \ldots x_n$ receding azimuthally with velocities $v_{-n} \ldots v_n$, with respect to $x = 0$.

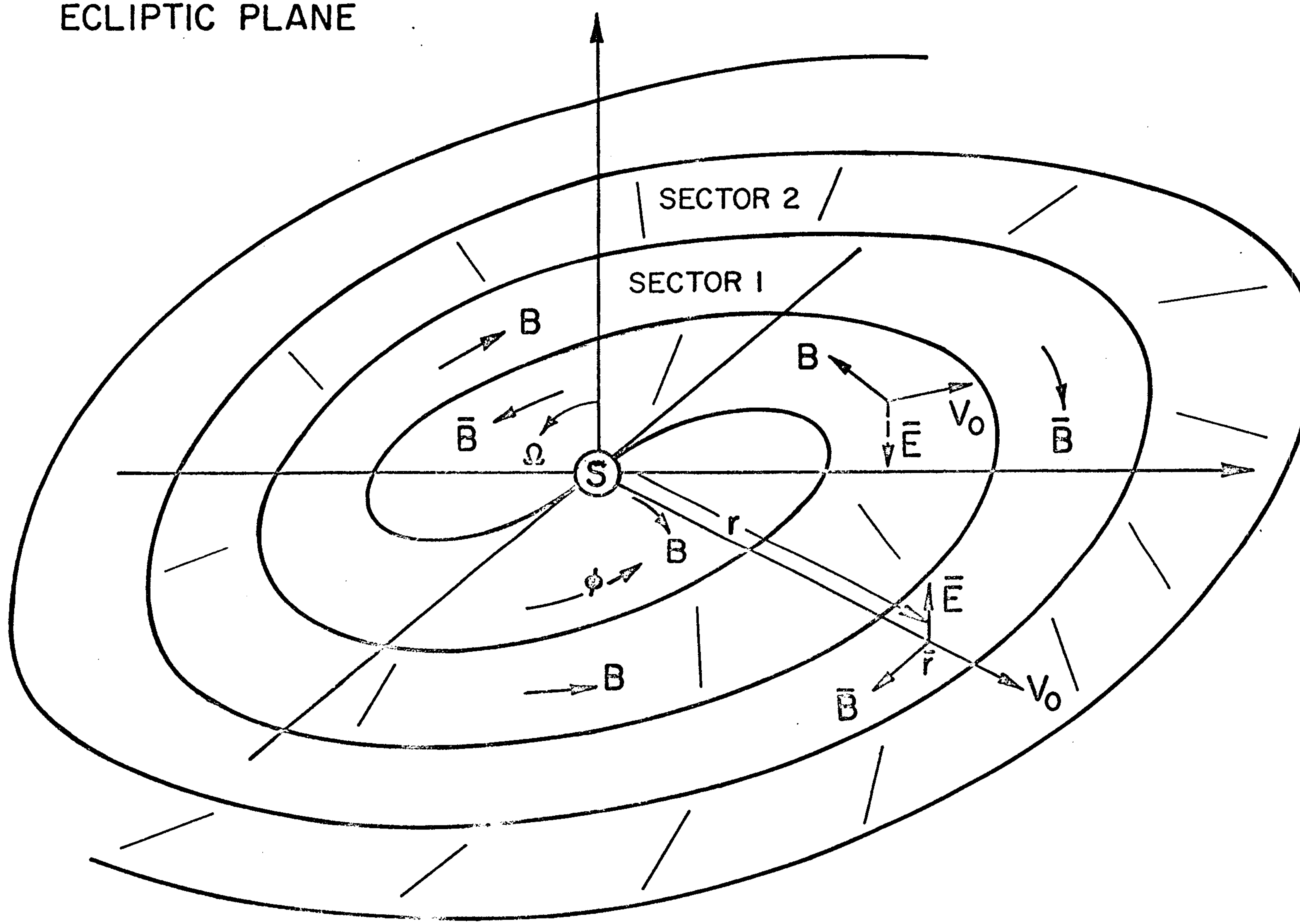

Fig. 1

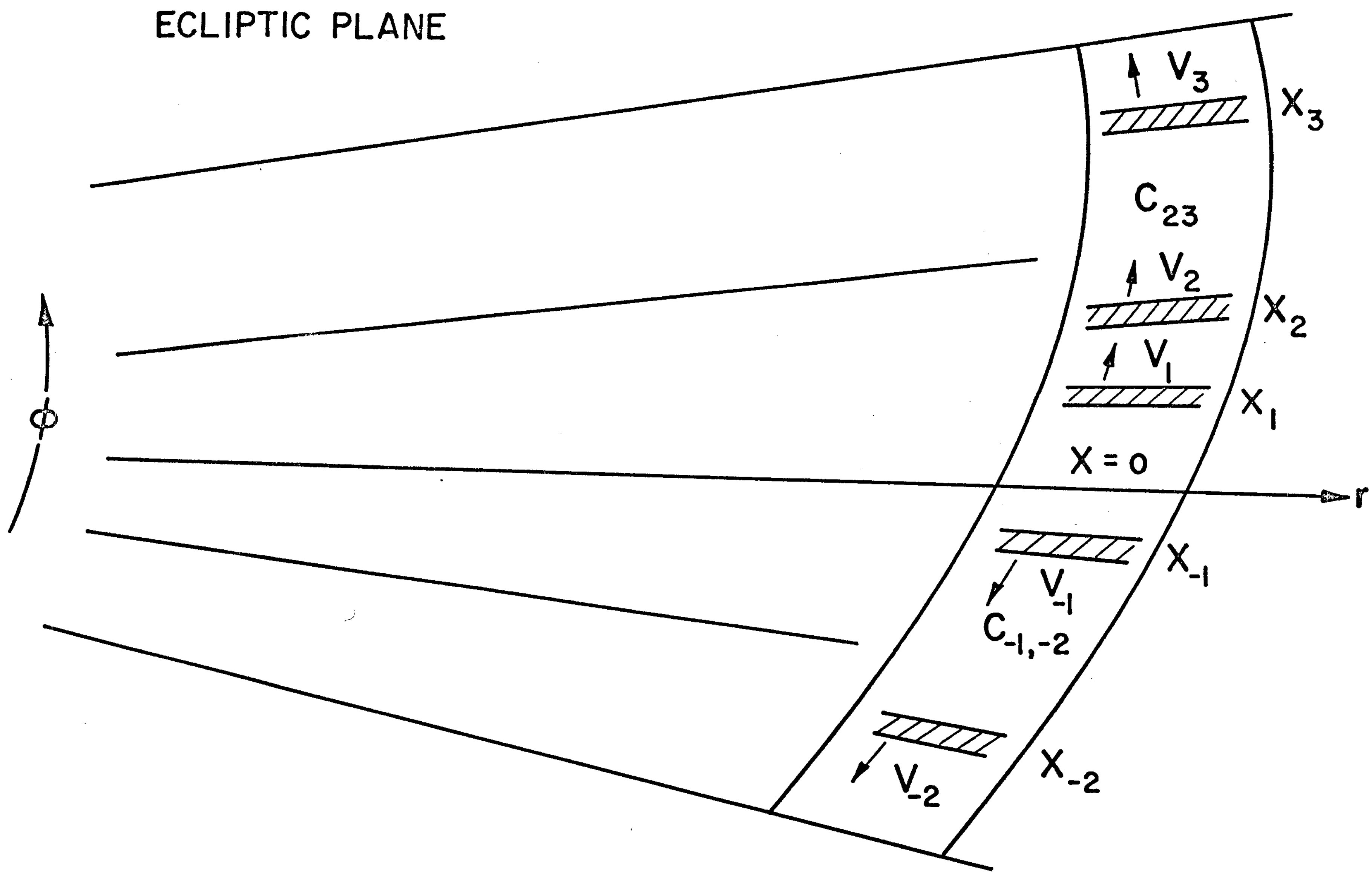

Fig. 2

APPENDIX A

In the following paragraphs we outline the steps leading from the Boltzmann equation to equations 4 and 5. To arrive at equation 4, we evaluate the integrals occuring in the Boltzmann equation, integrated over v = constant in momentum space:

$$\frac{\partial h}{\partial t} + \int v_i \frac{\partial f}{\partial x_i} v^2 d\Omega' + \int \frac{F_i}{m} \frac{\partial f}{\partial v_i} v^2 d\Omega' = \left.\frac{\partial h}{\partial t}\right)_c \tag{A1}$$

To facilitate the angular integration we choose spherical coordinates as the intependent variables in phase space - r, θ and ϕ representing a position and v θ' and ϕ' representing a velocity. The second term becomes

$$\int v_i \frac{\partial f}{\partial x_i} v^2 d\Omega' = \frac{\partial}{\partial x_i} \int v_i f v^2 d\Omega' = \frac{\partial}{\partial x_i} (h\langle v_i\rangle) \tag{A2}$$

since x_i and v_i are independent variables. The third term becomes

$$\int \frac{F_i}{m} \frac{\partial f}{\partial v_i} v^2 d\Omega' = \frac{1}{m} \int \frac{\partial}{\partial v_i} (f\ F_i)\ v^2 d\Omega' \tag{A3}$$

$$= \frac{1}{m} \int \left[\frac{1}{v^2} \frac{\partial}{\partial v} (v^2 F_v f) + \frac{1}{v\sin\theta'} \frac{\partial}{\partial \theta'} (\sin\theta' F_{\theta'} f) + \frac{1}{v\sin\theta'} \frac{\partial}{\partial \phi'} (F_{\phi'} f) \right] v^2 d\Omega'$$

since $\partial F_i/\partial v_i = 0$ for the Lorentz force. Since $d\Omega' = \sin\theta' d\theta' d\phi'$, the second and third terms become

$$\int v \left[\int_{\theta'=0}^{\pi} \frac{\partial}{\partial \theta'} (\sin\theta' F_{\theta'} f) d\theta' \right] d\phi' + \int v \left[\int_{\phi'=0}^{2\pi} \frac{\partial}{\partial \phi'} (F_{\phi'} f) d\phi' \right] d\theta' = 0$$

since the integrand in the first term vanishes and because F_ϕ, f must have the same value at $\phi' = 0$ and $\phi' = 2\pi$. Therefore equation A3 becomes

$$\int \frac{F_i}{m} \frac{\partial f}{\partial v_i} v^2 d\Omega' = \frac{1}{m} \int \frac{\partial}{\partial v} (v^2 F_v f)\ d\Omega' = \frac{1}{m} \frac{\partial}{\partial v} (<F_v>h)$$

and since only the electric field has a force component along the particle velocity,

$$<F_v> = \frac{q}{v} <E_i v_i>$$

Therefore,

$$\int \frac{F_i}{m} \frac{\partial f}{\partial v_i} v^2 d\Omega' = \frac{q}{m} E_i \frac{\partial}{\partial v} \left(\frac{h<v_i>}{v} \right)$$

so that equation A1 becomes

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_i} (h<v_i>) + \frac{q}{m} E_i \frac{\partial}{\partial v} \left(\frac{h<v_i>}{v}\right) = \left.\frac{\partial h}{\partial t}\right)_c \qquad (4)$$

To obtain equation 5 we evaluate the integrals occuring in the velocity moment of the Boltzmann equation over the shell, v = constant:

$$\int v_j \frac{\partial f}{\partial t} v^2 d\Omega' + \int v_j v_i \frac{\partial f}{\partial x_i} v^2 d\Omega' + \int v_j \frac{F_i}{m} \frac{\partial f}{\partial v_i} v^2 d\Omega' = \int v_j \left.\frac{\partial f}{\partial t}\right._c v^2 d\Omega' \qquad (A4)$$

Since v_j, v_i and t are independent variables, the first two terms become

$$\frac{\partial}{\partial t} (h<v_j>) + \frac{\partial}{\partial x_i} (h<v_j v_i>) = \frac{\partial}{\partial t} (h<v_i>) + \frac{\partial}{\partial x_i} [<v_i><v_j>h + S^{ij}]$$

with S^{ij} as defined in equation 5. Let us now consider the third term of equation A4. We have that

$$v_j F_i \frac{\partial f}{\partial v_i} = \frac{\partial}{\partial v_i}(v_j F_i f) - f v_j \frac{\partial F_i}{\partial v_i} - f F_i \delta_{ij} = \frac{\partial}{\partial v_i}(v_j F_i f) - F_j f$$

Therefore,

$$\int v_j \frac{F_i}{m}\frac{\partial f}{\partial v_i} v^2 d\Omega' = -\int \frac{F_j}{m} f v^2 d\Omega' + \int \frac{1}{mv^2}\frac{\partial}{\partial v}(v^2 F_v v_j f) v^2 d\Omega'$$

since the angular part of the divergence vanishes as for equation A3. Hence,

$$\int v_j \frac{F_i}{m}\frac{\partial f}{\partial v_i} v^2 d\Omega' = \frac{1}{m}\left[-h\langle F_j\rangle + \frac{\partial}{\partial v}(h\langle F_v v_j\rangle)\right]$$

$$= \frac{1}{m}\left[\frac{\partial}{\partial v}\left(\frac{\langle E_i v_i v_j\rangle}{v} h\right) - h\langle F_j\rangle\right] = \frac{E_i}{m}\frac{\partial}{\partial v}\left[\frac{h}{v}\langle v_i\rangle\langle v_j\rangle + \frac{S^{ij}}{v}\right] - \frac{h\langle F_j\rangle}{m}$$

Equation A4 therefore becomes

$$\frac{\partial}{\partial t}(h\langle v_j\rangle) + \frac{\partial}{\partial x_i}\left[\langle v_i\rangle\langle v_j\rangle h + S^{ij}\right] + \frac{qE_i}{m}\frac{\partial}{\partial v}\left[\frac{h}{v}\langle v_i\rangle\langle v_j\rangle + \frac{S^{ij}}{v}\right] - \frac{h\langle F_j\rangle}{m} = P_j$$

with P_j as defined in equation 5. The above equation may be rewritten as

$$h\left[\frac{\partial\langle v_j\rangle}{\partial t} + \langle v_i\rangle\frac{\partial\langle v_j\rangle}{\partial x_i} + \frac{qE_i\langle v_i\rangle}{mv}\frac{\partial\langle v_j\rangle}{\partial v}\right] + \langle v_j\rangle\left[\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_i}(h\langle v_i\rangle) + \frac{qE_i}{m}\frac{\partial}{\partial v}\left(\frac{h}{v}\langle v_i\rangle\right)\right]$$

$$+ \frac{\partial}{\partial x_i} S^{ij} + \frac{qE_i}{m}\frac{\partial}{\partial v}\left(\frac{S^{ij}}{v}\right) - \frac{h}{m}\langle F_j\rangle = P_j$$

Making use of equation 4, this finally becomes

$$h\frac{D\langle v_j\rangle}{Dt} = -\langle v_j\rangle \left.\frac{\partial h}{\partial t}\right)_c - \frac{\partial}{\partial x_i} S^{ij} - \frac{qE_i}{m}\frac{\partial}{\partial v}\left(\frac{S^{ij}}{v}\right) + \frac{h}{m}\langle F_j\rangle + P_j \qquad (5)$$

APPENDIX B

In the following paragraphs we consider the cosmic ray energy loss produced by scattering from field irregularities. In our transport equation for the energy spectrum, equation 10, this energy exchange is contained in the Fokker Planck coefficient, $<\Delta v>/\Delta t$, which was defined in terms of a field-free medium having a uniform cosmic ray distribution. Under such circumstances one is looking at the scattering without interference from field and gradient effects, and the arguments leading to the relaxation approximation, equation 12, indicate that the streaming velocity of the particle distribution h' approaches the solar wind velocity, no matter how the streaming velocity is initially prepared. A steady state description of the particles comprising h' is therefore that they stream with the solar wind.

The problem now reduces to considering a distribution of cosmic rays whose center of mass is at rest in an expanding configuration of irregular magnetic field. This expansion occurs normal to the direction of plasma flow, and we shall initially represent the expanding configuration of magnetic irregularities by the artifice of receding pistons. (See Figure II) The pistons are receding with a velocity proportional to their distance from the origin and any pair of pistons defines a cell in which cosmic rays are confined. To the extent that the particles represent an ideal gas, the particles within cell $c_{n,n+1}$ obey

$$(kT)V_{n,n+1}^{\frac{2}{m}} = \text{constant}, \quad \frac{1}{Kt}\frac{d(kT)}{dt} = -\frac{2}{mV_{n,n+1}}\frac{dV_{n,n+1}}{dt} \qquad \text{(B1)}$$

where m is the number of translational degrees of freedom for the particles, T is the particle temperature and $V_{n,n+1}$ is the volume of cell $c_{n,n+1}$

For the case of expansion in the ecliptic plane only,

$$V_{n,n+1} = A(\mathbf{X}_{n+1} - \mathbf{X}_n), \quad \frac{dV_{n,n+1}}{dt} = A(v_{n+1} - v_n) = A\,\frac{V_o}{r}\,(x_{n+1} - x_n)$$

Consequently, there is a uniform particle cooling rate,

$$\frac{1}{kT}\,\frac{d(kT)}{dt} = -\,\frac{2}{m}\,\frac{V_o}{r} \tag{B2}$$

in <u>all</u> cells, even though particles momentarily gain energy in distant cells when they rebound from the piston nearer the observer (at the origin). We may now remove the artifice of pistons and recall that we are dealing with a random, expanding magnetic structure for which the x_n are merely fiducial points having expansion velocities, $\frac{V_o x_n}{r}$. Rather than being strictly confined, particles now scatter randomly and migrate from one cell to another. However we have just seen that the cooling rate is the same in all cells and is independent of the cell size. Therefore as long as there is scattering from the expanding configuration, the particles cool according to equation B2. Equation B2 refers to the temperature of the distribution of particles, h'(v), which is related to the speed of the particles according to

$$kT = \frac{m}{3}\,(\langle v^2\rangle - \langle v\rangle^2) = \frac{m}{3}\,(v^2 - \langle v\rangle^2)$$

since we are considering particles initially of the same speed which are losing speed at the same rate. As long as $v^2 \gg \langle v\rangle^2$, equation B2 becomes

$$\frac{1}{v}\,\frac{dv}{dt} = -\,\frac{V_o}{mr} \tag{B3}$$

or in the language of equation 8,

$$P\ (v - \Delta v, \Delta v) = \delta(\Delta v + \frac{vV_o}{mr})$$

Therefore, the first Fokker Planck coefficient becomes

$$\langle \Delta v \rangle / \Delta t = \frac{-vV_o}{mr} \qquad \text{(B4)}$$

The $n^{\underline{th}}$ Fokker Planck coefficient becomes

$$\frac{\langle \Delta v^n \rangle}{\Delta t} = \left(\frac{vV_o}{mr} \right)^n \Delta t^{n-1} \longrightarrow 0 \text{ as } \Delta t \to 0.$$

Setting m = 3 and generalizing to the two dimensional expansion of the solar wind we arrive at

$$\frac{\langle \Delta v \rangle}{\Delta t} = -\frac{2vV_o}{3r} \qquad (18)$$

Our treatment of the cosmic ray cooling may appear questionable since our results are based upon a distribution of cosmic rays which is streaming outward at the solar wind velocity whereas the cosmic ray gas is istropic and hence stationary. The resolution of this discrepancy lies in the definition of the Fokker-Planck coefficient, $\langle \Delta v \rangle / \Delta t$ (equation 9). This coefficient measures the rate at which scattering changes the particle speed in the <u>absence</u> of fields and gradients and is therefore calculated for a test particle (or system of particles) placed in the scattering regime. After a time longer than the relaxation time, any such system of particles is streaming with the solar wind, but this system of particles remains the appropriate system for calculating $\langle \Delta v \rangle / \Delta t$. The fact that there is another process (diffusion) which creates an overall isotropy by introducing new particles to every volume element is therefore irrelevant. Diffusion causes the set of cosmic rays located within a given volume element at a particular instant to be isotropic but $\langle \Delta v \rangle / \Delta t$ concerns itself with what happens to this set of particles during the next Δt.